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Fuzzy Ideals and Fuzzy Filters in Core Regular Double Stone Algebras

Penki Surekha

Principal, Government Degree College (Men) Srikakulam

ABSTRACT

Core Regular Double Stone Algebras (CRDSAs) form a distinguished class of bounded distributive lattices endowed with pseudocomplementation and dual pseudocomplementation together with a unique core element. Although fuzzy ideals and fuzzy filters have been widely studied in lattices and several lattice-based algebras, comparatively little attention has been devoted to fuzzy structures adapted to CRDSAs. In this paper we introduce fuzzy ideals and fuzzy filters on CRDSAs and obtain α-cut characterizations linking fuzzy notions with their crisp counterparts. Motivated by the intrinsic double Stone operations, we also define *star-compatible* fuzzy ideals and *plus-compatible* fuzzy filters and derive basic consequences of these compatibility requirements. Prime and maximal fuzzy ideals are discussed via cut-set methods. Illustrative examples and graphical representations (TikZ and PGFPlots) are included.

Keywords: Core Regular Double Stone Algebra; Fuzzy ideal; Fuzzy filter; Star-compatible fuzzy ideal; Plus-compatible fuzzy filter; α-cut.
MSC 2020: 06D30; 03E72; 06B35.

INTRODUCTION

Fuzzy set theory introduced by Zadeh [4] has become a fundamental tool for handling uncertainty in algebraic structures. Since then, fuzzy ideals and fuzzy filters have been studied in distributive lattices, Boolean algebras, residuated lattices, MV-algebras and related systems. Core Regular Double Stone Algebras (CRDSAs) provide an algebraic framework extending Stone-type algebras by combining pseudocomplementation and dual pseudocomplementation with regularity and a distinguished core element.

The aims of this paper are: [label=()]

1. To define fuzzy ideals and fuzzy filters in CRDSAs,
2. To characterize these notions by means of α-cuts,
3. To introduce compatibility conditions reflecting the interaction with the operations * and $^+$,
4. To provide illustrative examples and figures.

2 Preliminaries

We recall basic notions needed in this work. For general background on lattices we refer to [1, 2].

Definition 2.1 *A double Stone algebra is an algebra*

$$L=(L,\vee,\wedge,{}^*,{}^+,0,1)$$

such that $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice, * is a pseudocomplementation, $^+$ is a dual pseudocomplementation, and the Stone identities hold:

$$x^*\vee x^{**}=1,\qquad x^+\wedge x^{++}=0\qquad (x\in L).$$

Definition 2.2 *A double Stone algebra L is called regular if for all $x,y\in L$,*

$$x^*=y^*\quad\text{and}\quad x^+=y^+\quad\Longrightarrow\quad x=y.$$

Definition 2.3 *Let L be a regular double Stone algebra. The centre of L is*

$$C(L)=\{x\in L: x^*=x^+\}.$$

Definition 2.4 *A regular double Stone algebra L is called a Core Regular Double Stone Algebra if there exists a unique element $k\in L$ (called the core element) such that*

$$k^*=0\qquad\text{and}\qquad k^+=1.$$

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Remark 2.5 *In general one does not have* $x^{**}=x$ *or* $x^{++}=x$ *for all* $x\in L$*; those equalities would force very strong conditions (e.g. Boolean behaviour). In this paper we work only with the standard axioms above.*

3 Fuzzy ideals

Throughout, a *fuzzy subset* of L means a map $\mu: L\to[0,1]$.

Definition 3.1 *A fuzzy subset* $\mu: L\to[0,1]$ *is called a fuzzy ideal of* L *if for all* $x,y\in L$, *[label=(I)]*

1. $\mu(x\vee y)\geq min\{\mu(x),\mu(y)\}$,
2. $x\leq y \Rightarrow \mu(x)\geq\mu(y)$.

Proposition 3.2 *If* μ *is a fuzzy ideal of* L, *then* $\mu(0)\geq\mu(x)$ *for all* $x\in L$.

Proof. Since $0\leq x$, condition (I2) gives $\mu(0)\geq\mu(x)$.

Example 3.3 *Let* $L=\{0<a<1\}$ *be the three-element chain (with the induced operations). Define*
$\mu(0)=1, \qquad \mu(a)=0.8, \qquad \mu(1)=0.5.$
Then μ is a fuzzy ideal of L.

4 Fuzzy filters

Definition 4.1 *A fuzzy subset* $\nu: L\to[0,1]$ *is called a fuzzy filter of* L *if for all* $x,y\in L$, *[label=(F)]*

1. $\nu(x\wedge y)\geq min\{\nu(x),\nu(y)\}$,
2. $x\leq y \Rightarrow \nu(x)\leq\nu(y)$.

5 α-cut characterizations

Definition 5.1 *Let* $\mu: L\to[0,1]$ *be a fuzzy subset and* $\alpha\in[0,1]$. *The* α*-cut of* μ *is*
$\mu_\alpha=\{x\in L: \mu(x)\geq\alpha\}.$

Theorem 5.2 *A fuzzy subset* μ *is a fuzzy ideal of* L *if and only if each nonempty* α*-cut* μ_α *is an (ordinary) ideal of* L.

Proof. Assume μ is a fuzzy ideal and let $\mu_\alpha\neq\varnothing$. If $x,y\in\mu_\alpha$, then $\mu(x)\geq\alpha$ and $\mu(y)\geq\alpha$, hence
$\mu(x\vee y)\geq min\{\mu(x),\mu(y)\}\geq\alpha,$
so $x\vee y\in\mu_\alpha$. If $z\leq x$ and $x\in\mu_\alpha$, then $\mu(z)\geq\mu(x)\geq\alpha$ by (I2), so $z\in\mu_\alpha$. Thus μ_α is an ideal.
Conversely, assume every nonempty μ_α is an ideal. For (I1), set $\alpha=min\{\mu(x),\mu(y)\}$. Then $x,y\in\mu_\alpha$, hence $x\vee y\in\mu_\alpha$, so $\mu(x\vee y)\geq\alpha$. For (I2), if $x\leq y$ and $\mu(y)=\beta$, then $y\in\mu_\beta$ implies $x\in\mu_\beta$ (downward closure), so $\mu(x)\geq\beta=\mu(y)$.

Theorem 5.3 *A fuzzy subset* ν *is a fuzzy filter of* L *if and only if each nonempty* α*-cut*
$\nu_\alpha=\{x\in L: \nu(x)\geq\alpha\}$
is a filter of L.

Proof. The proof is dual to Theorem 5.2.

6 Compatibility with the double Stone operations

The preceding definitions use only the lattice reduct of L. For CRDSA-oriented fuzzy theory, we introduce compatibility conditions involving * and $^+$.

Definition 6.1 *A fuzzy ideal* μ *of* L *is called star-compatible if*
$\mu(x^{**})=\mu(x) \qquad \text{forall } x\in L.$

Definition 6.2 *A fuzzy filter* ν *of* L *is called plus-compatible if*

$v(x^{++}) = v(x) \qquad \text{forall } x \in L.$

Theorem 6.3 *Let μ be a fuzzy ideal of L. Then μ is star-compatible if and only if for every $\alpha \in [0,1]$, the cut μ_α is closed under $x \mapsto x^{**}$, i.e.*
$x \in \mu_\alpha \;\Rightarrow\; x^{**} \in \mu_\alpha.$

Proof. If $\mu(x^{**}) = \mu(x)$ and $x \in \mu_\alpha$, then $\mu(x^{**}) = \mu(x) \geq \alpha$, hence $x^{**} \in \mu_\alpha$.
Conversely, assume every cut is closed under $x \mapsto x^{**}$. Fix $x \in L$ and let $\alpha = \mu(x)$. Then $x \in \mu_\alpha$, hence $x^{**} \in \mu_\alpha$, so $\mu(x^{**}) \geq \mu(x)$. Since $x \leq x^{**}$ holds in any pseudocomplemented distributive lattice, (I2) gives $\mu(x) \geq \mu(x^{**})$. Therefore $\mu(x^{**}) = \mu(x)$.

7 Prime and maximal fuzzy ideals

Definition 7.1 *A fuzzy ideal μ is called prime if for all $x, y \in L$,*
$\mu(x \wedge y) \leq max\{\mu(x), \mu(y)\}.$

Theorem 7.2 *A fuzzy ideal μ is prime if and only if every nonempty proper α-cut μ_α is a prime ideal of L.*
Proof. This follows from standard cut-set arguments: the inequality for μ translates into the usual primeness condition for μ_α, and conversely primeness of all relevant cuts yields the inequality by choosing $\alpha = \mu(x \wedge y)$.

Definition 7.3 *A fuzzy ideal μ is called maximal if $\mu \neq 1$ (the constant 1 function) and there is no fuzzy ideal η such that $\mu < \eta < 1$ (pointwise strict inequality at some point).*

Remark 7.4 *Further structure theorems for maximal fuzzy ideals typically depend on additional hypotheses or a specific chosen notion of maximality; here we restrict to this basic order-theoretic definition.*

8 Illustrations (TikZ and PGFPlots)

8.1 Hasse diagram (lattice picture)
The following figure is a standard four-element distributive lattice (diamond). It is included only as a *diagram template*. When presenting a specific CRDSA example, the diagram should match the verified order of that example.

8.2 Membership function plot
For the fuzzy ideal in Example 3.3, one may visualize the membership values as follows.

CONCLUSION

In this paper we introduced fuzzy ideals and fuzzy filters on Core Regular Double Stone Algebras and obtained α-cut characterizations. We also proposed star-compatible fuzzy ideals and plus-compatible fuzzy filters to connect fuzzy membership functions with the operators * and $^+$. Further directions include detailed study of star-/plus-compatible prime and maximal fuzzy ideals, fuzzy congruences on CRDSAs, and interactions with rough approximations and formal concept analysis.

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