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Coefficient Inequality for a New Class of Analytic Functions

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ABSTRACT

We introduce a new class of Convex Starlike analytic functions and its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes.

Keywords: Univalent functions, Convex functions, Starlike functions, Convex Starlike functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C45, 30C50

INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([8], [9]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [6] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and $a_2{}^2$ for the class $\mathcal{S}$, Fekete and Szegö[10] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (Chhichra[1], Babalola[7], Goel and Mehrok ([2],[3],[12]). Kaplan [4], Keogh and Merkes [5] proved several important results about these classes of analytic univalent functions.

Denote by $S_p{}^*(\varphi)$ the class of functions f analytic in $\mathbb{E}$ for which $\left(\frac{zf'(z)}{pf(z)}\right) \prec \varphi(z); z \in \mathbb{E}$.The class was defined and studied by Ali, Ravichandran, and Seenivasagan [11]. They obtained the Fekete-Szegő inequality for functions in the class $S_p{}^*(\varphi)$. The class $S_1{}^*(\varphi)$ coincides with the class $S^*(\varphi)$ discussed by Ma and Minda [13].

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n , z \in \mathcal{A}$ and satisfying the condition

$$Re \frac{\left((zh'(z)\right)'}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

We introduce the class of Convex Starlike functions as the functions $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) > 0, z \in \mathbb{E} \tag{1.5}$$

i.e. $\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) \prec \frac{1+z}{1-z}.$

We will denote this class by $\mathcal{KS}^*$.

We will denote by $\mathcal{KS}^*[A, B]$, the subclass of $\mathcal{KS}^*$ consisting of the functions $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) \prec\prec \frac{1+Az}{1+Bz}; -1 \le B \le A \le 1 \tag{1.6}$$

Symbol $\prec$ stands for subordination, which we define as follows:

p-valent functions: A function $f(z) \in \mathcal{A}_p$ is said to be a **p-valent function** in E if it assumes no value more than p times in E. We will denote by $\mathcal{KS}^*$, the class of functions $f(z) \in \mathcal{A}_p$ satisfying the condition $\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) \prec \frac{1+z}{1-z}.$

We will call these functions **p-valently convex starlike functions.**

We will denote by $\mathcal{KS}_p{}^*[A, B]$, the subclass of $\mathcal{KS}^*$ consisting of the functions $f(z) \in \mathcal{A}_p$ and satisfying the condition

$$\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) \prec \frac{1+z}{1-z}; -1 \le B \le A \le 1.$$

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in E$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n , w(0) = 0, |w(z)| < 1. \tag{1.7}$$

It is known that

$|d_1| \le 1, |d_2| \le 1 - |d_1|^2.$

MAIN RESULTS

Theorem 2.1: If $f(z) \in \mathcal{KS}^*$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{3} - \frac{\mu}{4}, \ if \mu \le \frac{8}{9} & (2.1.1) \\ \frac{1}{9}, if \ \frac{8}{9} \le \mu \le \frac{16}{9} & (2.1.2) \\ \frac{\mu}{4} - \frac{1}{3}, \ if \mu \ge \frac{16}{9} & (2.1.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{KS}^*$, we get

$$\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right)=\frac{1+w(z)}{1-w(z)}. \quad (2.1.4)$$

Expanding (2.1.4), we get

$$(1+8a_2z+27a_3z^2+--)=\ (1+4a_2z+9a_3z^2+--)\left(1+2c_1z\ +2\left(c_2+c_1^2\right)z^2+--\right) \quad (2.1.5)$$

Identifying the terms, we can easily obtain

$a_2=\frac{1}{2}c_1$ and $a_3\ =\frac{1}{9}c_2+\frac{1}{3}c_1^2$

Using these values, we can write

$$a_3-\mu a_2^2\ =\frac{1}{9}c_2\ +\left(\frac{1}{3}-\frac{\mu}{4}\right)c_1^2 \quad (2.1.6)$$

Taking absolute value, we get from (2.1.6)

$$|a_3-\mu a_2^2|\leq\frac{1}{9}|c_2|+\left|\frac{1}{3}-\frac{\mu}{4}\right||c_1|^2\ \leq\frac{1}{9}+\left(\left|\frac{1}{3}-\frac{\mu}{4}\right|-\frac{1}{9}\right)|c_1|^2 \quad (2.1.7)$$

Case I : $\mu\leq\frac{4}{3}$, (2.1.7) reduces to

$$|a_3-\mu a_2^2|\leq\frac{1}{9}+\left(\frac{2}{9}-\frac{\mu}{4}\right)|c_1|^2 \quad (2.1.8)$$

Subcase I(a) : $\mu\leq\frac{8}{9}$. We obtain from (2.1.8)

$$|a_3-\mu a_2^2|\leq\frac{1}{3}-\frac{\mu}{4} \quad (2.1.9)$$

Subcase I(b) $\mu\ \geq\frac{8}{9}$. We obtain from (2.1.8)

$$|a_3-\mu a_2^2|\leq\frac{1}{9}$$

Case II: $\mu\ \geq\frac{4}{3}$, we get

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{9}+\left(\frac{\mu}{4}-\frac{4}{9}\right)|c_1|^2$$

Subcase II(a) : $\mu\leq\frac{16}{9}$, we get

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{9}$$

Combining subcase I(b) and subcase II(a), we get

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{9}, if\ \frac{8}{9}\leq\mu\leq\frac{16}{9} \quad (2.1.10)$$

Subcase II(b) $\mu\ \geq\frac{16}{9}$

$$\left|a_3-\mu a_2^2\right|\leq\ \frac{\mu}{4}-\frac{1}{3} \quad (2.1.11)$$

This completes the theorem. The results are sharp. Extremal function for (2.1.1) and (2.1.3) is $f_1(z)=log\frac{z}{1-z}$. Extremal function for (2.1.2) is $f_2(z)=\frac{1}{2z}log\frac{1+z}{1-z}$.

Theorem 2.2: If $f(z)\in\mathcal{K}S^*[A,B]$, then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{A-2B}{9}-\frac{\mu(A-B)}{8}, & if\mu \le \frac{8(A-2B-1)}{9(A-B)} \quad (2.2.1) \\ \frac{1}{9}, if\ \frac{8(A-2B-1)}{9(A-B)} \le \mu \le \frac{8(A-2B+1)}{9(A-B)} & \quad (2.2.2) \\ \frac{\mu(A-B)}{8}-\frac{A-2B}{9}, & if\mu \ge \frac{8(A-2B+1)}{9(A-B)} \quad (2.2.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{KS}^*[A,B]$, we get

$$\left(\frac{\left\{z\left(zf'(z)\right)'\right\}'}{\left(zf'(z)\right)'}\right) = \frac{1+Aw(z)}{1-Aw(z)} \qquad (2.2.4)$$

Expanding (2.2.4), we get

$$(1+8a_2z+27a_3z^2+--) = (1+4a_2z+9a_3z^2+--)$$

$$(1+(A-B)\,c_1z+\,(A-B)\,(c_2-Bc_1^2)z^2+--) \qquad (2.2.5)$$

Identifying terms, we get

$a_2 = \frac{1}{4}(A-B)c_1$ and

$a_3 = \frac{1}{18}[(A-B)c_2+(A-B)\,(A-2B)\,c_1^2]$

Using these values, we can write

$$a_3-\mu a_2^2 = \frac{1}{18}(A-B)c_2+\frac{1}{18}(A-B)(A-2B)c_1^2-\mu\,(\frac{1}{16}(A-B)^2c_1^2)$$

This gives us

$$\frac{2}{(A-B)}\left|a_3-\mu a_2^2\right|$$

$$\le \frac{1}{9}|c_2|+\left|\frac{A-2B}{9}-\frac{\mu(A-B)}{8}\right|\left|c_1^2\right|$$

$$\le \frac{1}{9}+\left\{\left|\frac{A-2B}{9}-\frac{\mu(A-B)}{8}\right|-\frac{1}{9}\right\}\left|c_1^2\right| \qquad (2.2.6)$$

Case I : $\mu \le \frac{8(A-2B)}{9(A-B)}$, (2.2.6) becomes

$$\frac{2}{(A-B)}\left|a_3-\mu a_2^2\right| \le \frac{1}{9}+\left\{\frac{A-2B-1}{9}-\frac{\mu(A-B)}{8}\right\}\left|c_1^2\right| \qquad (2.2.7)$$

Subcase I (a): $\mu \le \frac{8(A-2B-1)}{9(A-B)}$. Using $\left|c_1^2\right| \le 1$, we get

$$\frac{2}{(A-B)}\left|a_3-\mu a_2^2\right|$$

$$\le \frac{1}{9}+\left\{\frac{A-2B-1}{9}-\frac{\mu(A-B)}{8}\right\}$$

$$= \frac{A-2B}{9}-\frac{\mu(A-B)}{8}$$

Subcase I(b) : $\mu \geq \frac{8(A-2B-1)}{9(A-B)}$. we get $\frac{2}{(A-B)}\left|a_3 - \mu a_2^2\right| \leq \frac{1}{9}$.

Case II: $\mu \leq \frac{8(A-2B)}{9(A-B)}$, (2.2.6) becomes

$$\frac{2}{(A-B)}\left|a_3 - \mu a_2^2\right|$$

$$\leq \frac{1}{9} + \left\{\frac{\mu(A-B)}{8} - \frac{A-2B+1}{9}\right\}\left|c_1^2\right|$$

Subcase II (a) : $\mu \leq \frac{8(A-2B+1)}{9(A-B)}$ we get $\frac{2}{(A-B)}\left|a_3 - \mu a_2^2\right| \leq \frac{1}{9}$

Subcase II (a) : $\mu \geq \frac{8(A-2B+1)}{9(A-B)}$

Using $\left|c_1^2\right| \leq 1$, we get

$$\frac{2}{(A-B)}\left|a_3 - \mu a_2^2\right|$$

$$\leq \frac{1}{9} + \left\{\frac{\mu(A-B)}{8} - \frac{A-2B+1}{9}\right\} = \frac{\mu(A-B)}{8} - \frac{A-2B}{9}.$$

This completes the theorem. The results are sharp.

Extremal function for (2.2.1) and (2.2.3) is given by

$$f'(z) = \begin{cases} \frac{\log(1+Bz)^{1/B}}{z}; if\ A = 2B \\ \frac{A(1+Bz)^{2B-A/B}}{(2B-A)z}; if\ A \neq 2B \end{cases}.$$

Extremal function for (2.2.2) is given by $(zf_2{}'(z))' = (1+Bz^2)^{B-A/2B}$.

Corollary 2.3: Putting $A = 1, B = -1$ in theorem we get

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{3} - \frac{\mu}{4},\ if\, \mu \leq \frac{8}{9} \\ \frac{1}{9}, if\ \frac{8}{9} \leq \mu \leq \frac{16}{9} \\ \frac{\mu}{4} - \frac{1}{3},\ if\, \mu \geq \frac{16}{9} \end{cases}$$

Which are the result of $\mathcal{KS}^*$.

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