EDUZONE: International Peer Reviewed/Refereed Multidisciplinary Journal (EIPRMJ), ISSN: 2319-5045
Volume 6, Issue 1, January-June, 2017, Impact Factor: 4.295, Available online at: www.eduzonejournal.com

Coefficient Inequality for a Certain Subclasses of Analytic Functions

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ABSTRACT

We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \le 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \le 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \le 2$ and $|a_3| \le 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu, if\ \mu \le 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \le \mu \le 1; \\ 4\mu - 3, if \mu \ge 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by $\mathfrak{C}$ and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B)=\left\{f(z)\in\mathcal{A};\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,z\in\mathbb{E}\right\} \quad (1.6)$$

$$\mathcal{K}(A,B)=\left\{f(z)\in\mathcal{A};\frac{(zf'(z))'}{f'(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,z\in\mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new class as $\left\{f(z)\in\mathcal{A};\frac{z\{(f'(z))^2+f(z)f''(z)\}}{f(z)f'(z)}\prec\frac{1+z}{1-z};z\in\mathbb{E}\right\}$ and we will denote this class as $S^*(f,f',f'')$.

We will also deal with two subclasses of $S^*(f,f',f'')$ defined as follows:

$$S^*(f,f',f'';A,B)=\left\{f(z)\in\mathcal{A};\frac{z\{(f'(z))^2+f(z)f''(z)\}}{f(z)f'(z)}\prec\frac{1+Az}{1+Bz};z\in\mathbb{E}\right\} \quad (1.8)$$

$$S^*(f,f',f'';A,B,\delta)=\left\{f(z)\in\mathcal{A};\frac{z\{(f'(z))^2+f(z)f''(z)\}}{f(z)f'(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^\delta;z\in\mathbb{E}\right\} \quad (1.9)$$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$; $z\in\mathbb{E}$ and we write $f(z)\prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z)=\sum_{n=1}^{\infty}c_nz^n, w(0)=0, |w(z)|<1$. (1.10)

It is known that $|c_1|\le 1, |c_2|\le 1-|c_1|^2$. (1.11)

PRELIMINARY LEMMAS

For $0<c<1$, we write $w(z)=\left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)}=1+2cz+2z^2+\cdots. \quad (2.1)$$

MAIN RESULTS

THEOREM 3.1: Let $f(z)\in S^*(f,f',f'')$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{19}{36}-\frac{4}{9}\mu, if\mu\le\frac{5}{8}; & (3.1)\\ \frac{1}{4}, if\ \frac{5}{8}\le\mu\le\frac{7}{4}; & (3.2)\\ \frac{4}{9}\mu-\frac{19}{36},\ \ if\ \mu\ge\frac{7}{4}. & (3.3)\end{cases}$$

The results are sharp.

Proof: By definition of $S^*(f,f',f'')$, we have

$$\frac{z\{(f'(z))^2+f(z)f''(z)\}}{f(z)f'(z)}=\frac{1+w(z)}{1-w(z)};w(z)\in\mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$(1+2a_2z+3a_3z^2+---)^2+(z+a_2z^2+a_3z^3+---)(2a_2+6a_3z+12a_4z^2+---)=(1+a_2z+a_3z^2+---)(1+2a_2z+3a_3z^2+---)(1+2c_1z+2(c_2+{c_1}^2)z^2+---)$.

$\{1+4a_2z+(6a_3+4a_2^2)z^2+---\}+\{2a_2z+(6a_3+2a_2^2)z^2+---\}=(1+3a_2z+(4a_3+2a_2^2)z^2+---)(1+2c_1z+2(c_2+{c_1}^2)z^2+---)$.

$$1+6a_2z+6(2a_3+a_2^2)z^2+---=1+(3a_2+2c_1)z+(4a_3+2a_2^2+6a_2c_1+2c_2+2{c_1}^2)z^2+--- \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2=\frac{2}{3}c_1 \quad (3.6)$$

$$a_3=\frac{1}{4}c_2+\frac{19}{36}{c_1}^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3-\mu a_2^2=\frac{1}{4}c_2+\left[\frac{19}{36}-\frac{4}{9}\mu\right]c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}|c_2| + \left|\frac{19}{36} - \frac{4}{9}\mu\right| |c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}(1 - |c_1|^2) + \left|\frac{19}{36} - \frac{4}{9}\mu\right| |c_1|^2 = \frac{1}{4} + \left\{\left|\frac{19}{36} - \frac{4}{9}\mu\right| - \frac{1}{4}\right\} |c_1|^2. \quad (3.10)$$

Case I: $\mu \leq \frac{19}{36}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\left(\frac{19}{36} - \frac{4}{9}\mu\right) - \frac{1}{4}\right\} |c_1|^2 = \frac{1}{4} + \left\{\frac{5}{18} - \frac{4}{9}\mu\right\} |c_1|^2. \quad (3.11)$$

Subcase I (a): $\mu \leq \frac{5}{8}$. Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\frac{5}{18} - \frac{4}{9}\mu\right\} = \frac{19}{36} - \frac{4}{9}\mu. \quad (3.12)$$

Subcase I (b): $\mu \geq \frac{5}{8}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} - \left\{\frac{4}{9}\mu - \frac{5}{18}\right\} |c_1|^2 \leq \frac{1}{4}. \quad (3.13)$$

Case II: $\mu \geq \frac{19}{36}$. Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\frac{4}{9}\mu - \frac{7}{9}\right\} |c_1|^2. \quad (3.14)$$

Subcase II (a): $\mu \leq \frac{7}{4}$. (3.14) takes the form $|a_3 - \mu a_2^2| \leq \frac{1}{4}$. (3.15)

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \text{ if } \frac{5}{8} \leq \mu \leq \frac{7}{4}. \quad (3.16)$$

Subcase II (b): $\mu \geq \frac{7}{4}$. Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{4}{9}\mu - \frac{19}{36}. \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by $f_1(z) = \sqrt{2\left\{\frac{1}{1-z} - \log(1 - z)\right\}}$.

Extremal function for (3.2) is defined by $f_2(z) = \sqrt{\log\left(\frac{1}{1-z^2}\right)}$.

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