EDUZONE: International Peer Reviewed/Refereed Multidisciplinary Journal (EIPRMJ), ISSN: 2319-5045
Volume 3, Issue 2, July-December, 2014, Available online at: www.eduzonejournal.com

A Study of Classical and Quantum Mechanics in Mechanical Engineering

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ABSTRACT

The relationship between classical and quantum theory is of central importance to the philosophy of physics, and any interpretation of quantum mechanics has to clarify it. Our discussion of this relationship is partly historical and conceptual, but mostly technical and mathematically rigorous, including several references. For example, the author has studied how certain intuitive ideas of the founders of quantum theory have fared in the light of current mathematical knowledge. One such idea that has certainly stood the test of time is Heisenberg's 'quantum-theoretical Umdeutung (reinterpretation) of classical observables', which lies at the basis of quantization theory. Similarly, Bohr's correspondence principle (in somewhat revised form) and Schrodinger's wave packets (or coherent states) continue to be of great importance in understanding classical behaviour from quantum mechanics. On the other hand, no consensus has been reached on the Copenhagen Interpretation, but in view of the parodies of it one typically finds in the literature we describe it in detail.

This combination of ideas and techniques does not quite resolve the measurement problem, but it does make the point that classicality results from the elimination of certain states and observables from quantum theory. Thus the classical world is not created by observation, but rather by the lack of it.

Keywords : Classical, Quantum Mechanics, physics, mechanical.

INTRODUCTION

Most modern physicists and philosophers would agree that a decent interpretation of quantum mechanics should fullfil at least two criteria. Firstly, it has to elucidate the physical meaning of its mathematical formalism and thereby secure the empirical content of the theory. This point (which we address only in a derivative way) was clearly recognized by all the founders of quantum theory. Secondly (and this is the subject of this paper), it has to explain at least the appearance of the classical world. As shown by our second quotation above, Planck saw the difficulty this poses, and as a first contribution he noted that the high-temperature limit of his formula for black-body radiation converged to the classical expression. Although Bohr believed that quantum mechanics should be interpreted through classical physics, among the founders of the theory he seems to have been unique in his lack of appreciation of the problem of deriving classical physics from quantum theory. Nonetheless, through his correspondence principle (which he proposed in order to address the first problem above rather than the second) Bohr made one of the most profound contributions to the issue. Heisenberg initially recognized the problem, but quite erroneously came to believe he had solved it in his renowned paper on the uncertainty relations. Einstein famously did not believe in the fundamental nature of quantum theory, whereas Schr¨odinger was well aware of the problem from the beginning, later highlighted the issue with his legendary cat, and at various stages in his career made important technical contributions towards its resolution. Ehrenfest stated the well-known theorem named after him. Von Neumann saw the difficulty, too, and addressed it by means of his well-known analysis of the measurement procedure in quantum mechanics.

On the assumption that quantum mechanics is universal and complete, we discuss three ways in which classical physics has so far been believed to emerge from quantum physics, namely in the limit $\sim \rightarrow 0$ of small Planck's constant (in a finite system), in the limit $N \rightarrow \infty$ of a large system with N degrees of freedom (at fixed ~), and through decoherence and consistent histories. The first limit is closely related to modern quantization theory and microlocal analysis, whereas the second involves methods of C ∗ -algebras and the concepts of superselection sectors and macroscopic observables. In these limits, the classical world does not emerge as a sharply defined objective reality, but rather as an approximate appearance relative to certain "classical" states and observables. Decoherence subsequently clarifies the role of such states, in that they are "einselected", i.e. robust against coupling to the environment. Furthermore, the nature of classical observables is elucidated by the fact that they typically define (approximately) consistent sets of histories.

The problem is actually even more acute than the founders of quantum theory foresaw. The experimental realization of Schr¨odinger's cat is nearer than most physicists would feel comfortable with (Leggett, 2002; Brezger et al., 2002; Chiorescu et al., 2003; Marshall et al., 2003; Devoret et al., 2004). Moreover, awkward superposition are by no means confined to physics laboratories: due to its chaotic motion, Saturn's moon Hyperion (which is about the size of New York) has been estimated to spread out all over its orbit within 20 years if treated as an isolated quantum-mechanical wave packet (Zurek & Paz, 1995). Furthermore, decoherence theorists have made the point that "measurement" is not only a procedure carried out by experimental physicists in their labs, but takes place in Nature all the time without any human intervention. On the conceptual side, parties as diverse as Bohm & Bell and their followers on the one hand and the quantum cosmologists on the other have argued that a "Heisenberg cut" between object and observer cannot possibly lie at the basis of a fundamental theory of physics.

These and other remarkable insights of the past few decades have drawn wide attention to the importance of the problem of interpreting quantum mechanics, and in particular of explaining classical physics from it.

We will discuss these ideas in more detail below, and indeed our discussion of the relationship between classical and quantum mechanics will be partly historical. However, other than that it will be technical and mathematically rigorous. For the problem at hand is so delicate that in this area sloppy mathematics is almost guaranteed to lead to unreliable physics and conceptual confusion (notwithstanding the undeniable success of poor man's math elsewhere in theoretical physics). Except for von Neumann, this was not the attitude of the pioneers of quantum mechanics; but while it has to be acknowledged that many of their ideas are still central to the current discussion, these ideas per se have not solved the problem. Thus we assume the reader to be familiar with the Hilbert space formalism of quantum mechanics, and for some parts of this paper (notably Section 6 and parts of Section 4) also with the basic theory of C
∗-algebras and its applications to quantum theory. In addition, some previous encounter with the conceptual problems of quantum theory would be helpful.

Which ideas have solved the problem of explaining the appearance of the classical world from quantum theory? In our opinion, none have, although since the founding days of quantum mechanics a number of new ideas have been proposed that almost certainly will play a role in the eventual resolution, should it ever be found. These ideas surely include:

• The limit ~ → 0 of small Planck's constant (coming of age with the mathematical field of micro local analysis);

• The limit N → ∞ of a large system with N degrees of freedom (studied in a serious only way after the emergence of C∗ -algebraic methods);

• Decoherence and consistent histories.

Mathematically, the second limit may be seen as a special case of the first, though the underlying physical situation is of course quite different. In any case, after a detailed analysis our conclusion will be that none of these ideas in isolation is capable of explaining the classical world, but that there is some hope that by combining all three of them, one might do so in the future.

Because of the fact that the subject matter of this review is unfinished business, to date one may adopt a number of internally consistent but mutually incompatible philosophical stances on the relationship between classical and quantum theory. Two extreme ones, which are always useful to keep in mind whether one holds one of them or not, are:

1. Quantum theory is fundamental and universally valid, and the classical world has only “relative” or “perspectival” existence.

2. Quantum theory is an approximate and derived theory, possibly false, and the classical world exists absolutely.

An example of a position that our modern understanding of the measurement problem9 has rendered internally inconsistent is:

3. Quantum theory is fundamental and universally valid, and (yet) the classical world exists absolutely.

CLASSICAL MECHANICS

It is a physical theory describing the motion of macroscopic objects, from projectiles to parts of machinery, and astronomical objects, such as spacecraft, planets, stars, and galaxies. For objects governed by classical mechanics, if the present state is known, it is possible to predict how it will move in the future (determinism), and how it has moved in the past (reversibility).

The earliest development of classical mechanics is often referred to as Newtonian mechanics. It consists of the physical concepts based on foundational works of Sir Isaac Newton, and the mathematical methods invented by Gottfried Wilhelm Leibniz, Joseph-Louis Lagrange, Leonhard Euler, and other contemporaries, in the 17th century to describe the motion of bodies under the influence of a system of forces. Later, more abstract methods were developed, leading to the reformulations of classical mechanics known as Lagrangian mechanics and Hamiltonian mechanics. These advances, made predominantly in the 18th and 19th centuries, extend substantially beyond earlier works, particularly through their use of analytical mechanics. They are, with some modification, also used in all areas of modern physics.

Classical mechanics provides extremely accurate results when studying large objects that are not extremely massive and speeds not approaching the speed of light. When the objects being examined have about the size of an atom diameter, it becomes necessary to introduce the other major sub-field of mechanics: quantum mechanics. To describe velocities that are not small compared to the speed of light, special relativity is needed. In cases where objects become extremely massive, general relativity becomes applicable. However, a number of modern sources do include relativistic mechanics in classical physics, which in their view represents classical mechanics in its most developed and accurate form.

The following introduces the basic concepts of classical mechanics. For simplicity, it often models real-world objects as point particles (objects with negligible size). The motion of a point particle is characterized by a small number of parameters: its position, mass, and the forces applied to it. Each of these parameters is discussed in turn.

In reality, the kind of objects that classical mechanics can describe always have a non-zero size. (The physics of very small particles, such as the electron, is more accurately described by quantum mechanics.) Objects with non-zero size have more complicated behavior than hypothetical point particles, because of the additional degrees of freedom, e.g., a baseball can spin while it is moving. However, the results for point particles can be used to study such objects by treating them as composite objects, made of a large number of collectively acting point particles. The center of mass of a composite object behaves like a point particle.

Classical mechanics uses common sense notions of how matter and forces exist and interact. It assumes that matter and energy have definite, knowable attributes such as location in space and speed. Non-relativistic mechanics also assumes that forces act instantaneously (see also Action at a distance).

QUANTUM MECHANICS

It is a fundamental theory in physics that provides a description of the physical properties of nature at the scale of atoms and subatomic particles:1.1 It is the foundation of all quantum physics including quantum chemistry, quantum field theory, quantum technology, and quantum information science.

Classical physics, the collection of theories that existed before the advent of quantum mechanics, describes many aspects of nature at an ordinary (macroscopic) scale, but is not sufficient for describing them at small (atomic and subatomic) scales. Most theories in classical physics can be derived from quantum mechanics as an approximation valid at large (macroscopic) scale.

Quantum mechanics differs from classical physics in that energy, momentum, angular momentum, and other quantities of a bound system are restricted to discrete values (quantization), objects have characteristics of both particles and waves (wave–particle duality), and there are limits to how accurately the value of a physical quantity can be predicted prior to its measurement, given a complete set of initial conditions (the uncertainty principle).

Quantum mechanics arose gradually from theories to explain observations which could not be reconciled with classical physics, such as Max Planck's solution in 1900 to the black-body radiation problem, and the correspondence between energy and frequency in Albert Einstein's 1905 paper which explained the photoelectric effect. These early attempts to understand microscopic phenomena, now known as the "old quantum theory", led to the full development of quantum mechanics in the mid-1920s by Niels Bohr, Erwin Schrödinger, Werner Heisenberg, Max Born, Paul Dirac and others. The modern theory is formulated in various specially developed mathematical formalisms. In one of them, a mathematical entity called the wave function provides information, in the form of probability amplitudes, about what measurements of a particle's energy, momentum, and other physical properties may yield.

PLANCK AND EINSTEIN

The relationship between classical physics and quantum theory is so subtle and confusing that historians and physicists cannot even agree about the precise way the classical gave way to the quantum!As Darrigol (2001) puts it: ‘During the past twenty years, historians [and physicists] have disagreed over the meaning of the quanta which Max Planck introduced in his black-body theory of 1900. The source of this confusion is the publication of Thomas Kuhn’s [(1978)] iconoclastic thesis that Planck did not mean his energy quanta to express a quantum discontinuity.’

As is well known (cf. Mehra & Rechenberg, 1982a, etc.), Planck initially derived Wien’s law forblackbody radiation in the context of his (i.e. Planck’s) program of establishing the absolute nature of irreversibility (competing with Boltzmann’s probabilistic approach, which eventually triumphed).

When new high-precision measurements in October 1900 turned out to refute Wien’s law, Planck firstguessed his famous expression

$E\nu/N\nu = h\nu/(eh\nu/kT - 1)$

for the correct law, en passant introducing two new constants of nature h and k,19 and subsequently, on December 14, 1900, presented a theoretical derivation of his law in which he allegedly introduced the idea that the energy of the resonators making up his black body was quantized in units of εν = hν (where ν is the frequency of a given resonator). This derivation is generally seen as the birth of quantum theory, with the associated date of birth just mentioned.

However, it is clear by now (Kuhn, 1978; Darrigol, 1992, 2001; Carson, 2000; Brush, 2002) that Planck was at best agnostic about the energy of his resonators, and at worst assigned them a continuous energy spectrum. Technically, in the particular derivation of his empirical law that eventually turned out to lead to the desired result (which relied on Boltzmann's concept of entropy),20 Planck had to count the number of ways a given amount of energy Eν could be distributed over a given number of resonators Nν at frequency ν. This number is, of course, infinite, hence in order to find a finite answer Planck followed Boltzmann in breaking up Eν into a large number Aν of portions of identical size εν, so that Aνεν = Eν. 21 Now, as we all know, whereas Boltzmann let εν → 0 at the end of his corresponding calculation for a gas, Planck discovered that his empirical blackbody law emerged if he assumed the relation εν = hν.

However, this postulate did not imply that Planck quantized the energy of his resonators. In fact, in his definition of a given distribution he counted the number of resonators with energy between say (k − 1)εν and kεν (for some k ∈ N), as Boltzmann did in an analogous way for a gas, rather than the number of resonators with energy kεν, as most physicists came to interpret his procedure. More generally, there is overwhelming textual evidence that Planck himself by no means believed or implied that he had quantized energy; for one thing, in his Nobel Prize Lecture in 1920 he attributed the correct interpretation of the energy-quanta εν to Einstein. Indeed, the modern understanding

of the earliest phase of quantum theory is that it was Einstein rather than Planck who, during the period 1900–1905, clearly realized that Planck's radiation law marked a break with classical physics (B¨uttner, Renn, & Schemmel, 2003). This insight, then, led Einstein to the quantization of energy. This he did in a twofold way, both in connection with Planck's resonators - interpreted by Einstein as harmonic oscillators in the modern way - and, in a closely related move, through his concept of a photon. Although Planck of course introduced the constant named after him, and as such is the founding father of the theory characterized by ~, it is the introduction of the photon that made Einstein at least the mother of quantum theory. Einstein himself may well have regarded the photon as his most revolutionary discovery, for what he wrote about his pertinent paper is not matched in self-confidence by anything he said about relativity: 'Sie handelt ¨uber die Strahlung und die energetischen Eigenschaften des Lichtes und ist sehr revolution¨ar.'This makes it clear that Einstein specifically discussed the nature of radiation, so that neither Planck nor Einstein in fact claimed that quantum theory amounted to the 'discretization of nature'.

BOHR

Bohr's brilliant model of the atom reinforced his idea that quantum theory was a theory of quanta. Since this model simultaneously highlighted the clash between classical and quantum physics and carried the germ of a resolution of this conflict through Bohr's equally brilliant correspondence principle, it is worth saying a few words about it here. Bohr's atomic model addressed the radiative instability of Rutherford's solar-system-style atom: according to the electrodynamics of Lorentz, an accelerating electron should radiate, and since the envisaged circular or elliptical motion of an electron around the nucleus is a special case of an accelerated motion, the electron should continuously lose energy and spiral towards the nucleus.26 Bohr countered this instability by three simultaneous moves, each of striking originality:

1. He introduced a quantization condition that singled out only a discrete number of allowed electronic orbits (which subsequently were to be described using classical mechanics, for example, in Bohr's calculation of the Rydberg constant R).

2. He replaced the emission of continuous radiation called for by Lorentz by quantum jumps with unpredictable destinations taking place at unpredictable moments, during which the atom emits light with energy equal to the energy difference of the orbits between which the electron jumps.

3. He prevented the collapse of the atom through such quantum jumps by introducing the notion of ground state, below which no electron could fall.

With these postulates, for which at the time there existed no foundation whatsoever, Bohr explained the spectrum of the hydrogen atom, including an amazingly accurate calculation of R. Moreover, he proposed what was destined to be the key guiding principle in the search for quantum mechanics in the coming decade, viz. the correspondence principle

In general, there is no relation between the energy that an electron loses during a particular quantum jump and the energy it would have radiated classically (i.e. according to Lorentz) in the orbit it revolves around preceding this jump. Indeed, in the ground state it cannot radiate through quantum jumps at all, whereas according to classical electrodynamics it should radiate all the time. However, Bohr saw that in the opposite case of very wide orbits (i.e. those having very large principal quantum numbers n), the frequency $\nu = (En - En-1)/h$ (with $En = -R/n2$) of the emitted radiation approximately corresponds to the frequency of the lowest harmonic of the classical theory, applied to electron motion in the initial orbit. Moreover, the measured intensity of the associated spectral line (which theoretically should be related to the probability of the quantum jump, a quantity out of the reach of early quantum theory), similarly turned out to be given by classical electrodynamics. This property, which in simple cases could be verified either by explicit computation or by experiment, became a guiding principle in situations where it could not be verified, and was sometimes even extended to low quantum numbers, especially when the classical theory predicted selection rules.

It should be emphasized that Bohr's correspondence principle was concerned with the properties of radiation, rather than with the mechanical orbits themselves. This is not quite the same as what is usually called the correspondence principle in the modern literature.30 In fact, although also this modern correspondence principle has a certain range of validity (as we shall see in detail in Section 5), Bohr never endorsed anything like that, and is even on record as opposing such a principle:

HEISENBERG

Heisenberg's (1925) paper Uber die quantentheoretische Umdeutung kinematischer und mechanischer ¨ Beziehungen32 is generally seen as a turning point in the development of quantum mechanics. Even A. Pais, no friend of Heisenberg's, conceded that Heisenberg's paper marked 'one of the great jumps - perhaps the greatest - in the development of twentieth century physics.' What did Heisenberg actually accomplish? This question is particularly interesting from the perspective of our theme.

At the time, atomic physics was in a state of crisis, to which various camps responded in different ways. Bohr's approach might best be described as damage control: his quantum theory was a hybrid of classical mechanics adjusted by means of ad hoc quantization rules, whilst keeping electrodynamics classical at all cost. Einstein, who had been the first physicist to recognize the need to quantize classical electrodynamics, in the light of his triumph with General Relativity nonetheless dreamt of a classical field theory with singular solutions as the ultimate explanation of quantum phenomena. Born led the radical camp, which included Pauli: he saw the need for an entirely new mechanics replacing classical mechanics, which was to be based on discrete quantities satisfying difference equations.

It was Heisenberg's genius to interpolate between Bohr and Born. The meaning of his Umdeutung was to keep the classical equations of motion, whilst reinterpreting the mathematical symbols occurring therein as (what were later recognized to be) matrices. Thus his Umdeutung x 7→ a(n, m) was a precursor of what now would be called a quantization map f 7→ Q~(f), where f is a classical observable, i.e. a function on phase space, and Q~(f) is a quantum mechanical observable, in the sense of an operator on a Hilbert space or, more abstractly, an element of some C ∗ -algebra. As Heisenberg recognized, this move implies the noncommutativity of the quantum mechanical observables; it is this, rather than something like a "quantum postulate" that is the defining characteristic of quantum mechanics. Indeed, most later work on quantum physics and practically all considerations on the connection between classical and quantum rely on Heisenberg's idea of Umdeutung. This even applies to the mathematical formalism as a whole.

We here use the term "observable" in a loose way. It is now well recognized that Heisenberg's claim that his formalism could be physically interpreted as the replacement of atomic orbits by observable quantities was a red herring, inspired by his discussions with Pauli. In fact, in quantum mechanics any mechanical quantity has to be "reinterpreted", whether or not it is observable. As Heisenberg (1969) recalls, Einstein reprimanded him for the illusion that physics admits an a priori notion of an observable, and explained that a theory determines what can be observed. Rethinking the issue of observability then led Heisenberg to his second major contribution to quantum mechanics, namely his uncertainty relations.

These relations were Heisenberg's own answer to the quote opening this paper. Indeed, matrix mechanics was initially an extremely abstract and formal scheme, which lacked not only any visualization but also the concept of a state (see below). Although these features were initially quite to the liking of Born, Heisenberg, Pauli, and Jordan, the success of Schr¨odinger's work forced them to renege on their radical stance, and look for a semiclassical picture supporting their mathematics; this was a considerable U-turn found such a picture, claiming that his uncertainty relations provided the 'intuitive content of the quantum theoretical kinematics and mechanics' (as his paper was called). His idea was that the classical world emerged from quantum mechanics through observation: 'The trajectory only comes into existence because we observe it.' This idea was to become extremely influential, and could be regarded as the origin of stance 1 in the Introduction.

THE LIMIT ~ → 0

It was recognized at an early stage that the limit ~ → 0 of Planck's constant going to zero should play a role in the explanation of the classical world from quantum theory. Strictly speaking, ~ is a dimensionful constant, but in practice one studies the semiclassical regime of a given quantum theory by forming a dimensionless combination of ~ and other parameters; this combination then re-enters the theory as if it were a dimensionless version of ~ that can indeed be varied. The oldest example is Planck's radiation formula (2.1), with temperature T as the pertinent variable. Indeed, the observation of Einstein (1905) and Planck (1906) that in the limit ~ν/kT → 0 this formula converges to the classical equipartition law Eν/Nν = kT may well be the first use of the ~ → 0 limit of quantum theory.

Another example is the Schr¨odinger equation (2.3) with Hamiltonian H = − ~ 2 2m ∆x + V (x), where m is the mass of the pertinent particle. Here one may pass to dimensionless parameters by introducing an energy scale ǫ typical of H, like ǫ = supx |V (x)|, as well as a typical length scale λ, such as λ = ǫ/ supx |∇V (x)| (if these quantities are finite). In terms of the dimensionless variable ˜x = x/λ, the rescaled Hamiltonian H˜ = H/ǫ is then dimensionless and equal to H˜ = −~˜2∆x˜ + V˜ (˜x), where ~˜ = ~/λ√ 2mǫ and V˜ (˜x) = V (λx˜)/ǫ. Here ~˜ is dimensionless, and one might study the regime where it is small (Gustafson & Sigal, 2003). Our last example will occur in the theory of large quantum systems, treated in the next Section. In what follows, whenever it is considered variable ~ will denote such a dimensionless version of Planck's constant.

Although, as we will argue, the limit ~ → 0 cannot by itself explain the classical world, it does give rise to a number of truly pleasing mathematical results. These, in turn, render almost inescapable the conclusion that the limit in question is indeed a relevant one for the recovery of classical physics from quantum theory. Thus the present section is meant to be a catalogue of those pleasantries that might be of direct interest to researchers in the foundations of quantum theory.

There is another, more technical use of the ~ → 0 limit, which is to perform computations in quantum mechanics by approximating the time-evolution of states and observables in terms of associated classical objects. This endeavour is known as semiclassical analysis. Mathematically, this use of the ~ → 0 limit is closely related to the goal of recovering classical mechanics from quantum mechanics, but conceptually the matter is quite different. We will attempt to bring the pertinent differences out in what follows.

CONCLUSION

Let us state the main points of the study.

1. All the calculations must be performed in the space with the Minkowski metric. This condition is important in the field theories with a high group symmetries (such as the theories of the Yang-Mills type) since for such theories one has yet not been able to perform adequately the analytic continuation into the Euclidean region. (The fact that a theory must satisfy certain conditions upon analytic continuation in time is clear from) Apart from that, in the pseudo-Euclidean metric one is able to take into account external conditions with nontrivial time dependence without any restrictions.

2. The quantization can be performed without the transition to the canonical formalism (see Sec.2), remaining in the Lagrange formalism which is a more natural formalism for relativistic field theories.

3. In obtaining the contributions to the functional integrals onle the exact solutions of the equation of motion must be taken into account.

4. The contributions to functional integrals are found by variation of the classical nonrenormalized action, which simplifies the calculations considerably. This important feature of our approach is discussed in forthcoming publications in the light of the phenomenon of spontaneous symmetry breaking.

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